Hybrid Data Tomography: Winter School and Workshop

On Thursday and Friday, January 25–26, 2018, participants of the winter school are offered to work in small groups of 2-3 people on a project from suggested or their own topics. Proposed topics include a short description and a list of ideas to explore. However, participants are not required to answer all suggested questions. Furthermore, the projects are open-ended and participants are encouraged to explore any aspects which appeal to them, with the project descriptions providing ideas. Theoretical or computational studies on the chosen topics should be documented in a short report (5-10 pages) and handed in at the latest on February 2, 2018. For implementing computational tasks, sample codes written in Python using the library FEniCS will be provided at the winter school. To get started with FEniCS, follow instructions at `https://fenicsproject.org/download/`. However, participants are welcome to use any programming language they wish.

Keywords: Jacobian to Solutions

Project 1: Hybrid Data Tomography

Assume, that the electrical potential u and conductivity σ satisfy the electrostatic equation

$$\begin{aligned} \operatorname{div}(\sigma \nabla u) &= 0\,, \quad \text{in } \Omega\,, \\ u|_{\partial\Omega} &= g\,, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, $n \geq 2$. The goal of this exercise is to investigate theoretically and numerically the condition

$$\det[\nabla u_1 \dots \nabla u_n](x) > 0\,, \quad \text{for all } x \in \Omega\,, \tag{2}$$

for the solutions $u_1, \dots, u_n$ of (1) on the corresponding boundary functions $g_1, \dots, g_n$.

(a) Explain the relevance of condition (2) for inverse problems of AET/CDII/MREIT [1–3]. Why this condition is important?

(b) Investigate condition (2) analytically for constant conductivity $\sigma = 1$ for the domain Ω given by a unit disk in 2D, i.e., $\Omega := \{(r, \theta) \in [0, 1) \times [0, 2\pi]\}$. Does it hold for the boundary conditions defined by the identity mappings $g_1(x) = x_1$ and $g_2(x) = x_2$ (in polar coordinates: $g_1(r, \theta) = \cos(\theta)$ and $g_2(r, \theta) = \sin(\theta)$). How about the boundary conditions $g_1(r, \theta) = \cos(2\theta)$ and $g_2(r, \theta) = \sin(2\theta)$? Solve the problem (1) analytically for given boundary conditions and calculate (2).

(c) Explore the possibility of constructing the boundary conditions $g_1, \ldots, g_n$ in 2D and 3D, which would satisfy (2)? Does it hold for Ω given by a unit ball in 3D with $g_1(x) = x_1$, $g_2(x) = x_2$, $g_3(x) = x_3$? What is the difference between 2D and 3D?

(d) Give computational examples in 2D or 3D for non-uniform conductivity, where this condition is satisfied and not satisfied. Download a Python code with the forward solver for 2D using the library FEniCS from organizers of the winter school.

(e) Consider the partial data case, i.e., when the boundary $\partial\Omega$ is partly accessible for electric measurements g. In this case, the boundary functions are defined as follows: $g \neq 0$ on $\Gamma_1 \subset \partial\Omega$ and $g = 0$ on $\Gamma_0 = \partial\Omega \setminus \Gamma_1$. Investigate condition (2) analytically for constant conductivity $\sigma = 1$ for the domain Ω given by a square in 2D, i.e., $\Omega := \{(x_1, x_2) \in [0, \pi] \times [0, \pi]\}$ and non-zero boundary conditions $g_i(x) = \sin(ix)$, $i = 1, 2, 3$, on $\Gamma_1 := \{(x_1, x_2) \in [0, \pi] \times \{\pi\}\}$. Download a Python code with the forward solver for 2D using the library FEniCS from organizers of the winter school. Solve the problem numerically and plot (2) to confirm analytical observations.

References

[1] G. S. Alberti and Y. Capdeboscq. Lectures on elliptic methods for hybrid inverse problems. *SAM Report*, 2016.

[2] P. Kuchment. Mathematics of hybrid imaging: A brief review. *The Mathematical Legacy of Leon Ehrenpreis*, 16:183–208, 2012.

[3] T. Widlak and O. Scherzer. Hybrid tomography for conductivity imaging. *Inverse Problems*, 28(8):084008, 2012.