Hybrid Data Tomography: Winter School and Workshop

On Thursday and Friday, January 25–26, 2018, participants of the winter school are offered to work in small groups of 2-3 people on a project from suggested or their own topics. Proposed topics include a short description and a list of ideas to explore. However, participants are not required to answer all suggested questions. Furthermore, the projects are open-ended and participants are encouraged to explore any aspects which appeal to them, with the project descriptions providing ideas. Theoretical or computational studies on the chosen topics should be documented in a short report (5-10 pages) and handed in at the latest on February 2, 2018. For implementing computational tasks, sample codes written in Python using the library FEniCS will be provided at the winter school. To get started with FEniCS, follow instructions at `https://fenicsproject.org/download/`. However, participants are welcome to use any programming language they wish.

Keywords: Inverse Problems, Parameter Identification, Electrical Conductivity, Landweber Iteration

Project 2: Hybrid Data Tomography

Assume, that the electrical potential u and conductivity σ satisfy the electrostatic equation

$$\begin{aligned} \operatorname{div}(\sigma \nabla u) &= 0\,, \quad \text{in } \Omega\,, \\ u|_{\partial\Omega} &= g\,, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $N \geq 2$. Consider the inverse problem of Acousto-Electrical Tomography (AET), which aims at identifying the unknown conductivity σ from internal data of the power density [1, 5, 6].

(a) Download a Python code with the forward solver using the library FEniCS from organizers of the winter school. Simulate and plot the interior data of the power density by computing

$$E_i(\sigma) := \sigma |\nabla u_i(\sigma)|^2\,, \tag{2}$$

where u_i is a solution of (1) from different boundary functions g_i, $i = 1, \ldots, M$, and predefined conductivity σ of your choice.

(b) The inverse problem AET can be written in the operator form

$$F(\sigma) = E\,, \tag{3}$$

where the nonlinear operator is introduced by

$$F : D(F) := \left\{\sigma \in H^2(\Omega) \,|\, \sigma \geq \underline{\sigma} > 0\right\} \to L^2(\Omega), \qquad \sigma \mapsto E(\sigma)\,. \tag{4}$$

The problem can be solved using one of iterative regularization methods [3, 4], which typically require computation of the Fréchet derivative. Assume that operator F is Fréchet differentiable. Show that its Fréchet derivative is given by

$$F'(\sigma)h = h\,|\nabla u(\sigma)|^2 + 2\,\sigma \nabla u(\sigma) \cdot \nabla(u'(\sigma)h)\,, \tag{5}$$

where $u'(\sigma)h$ is a Fréchet derivative of the electric potential u given as the unique solution of the variational problem

$$\int_\Omega \sigma \nabla(u'(\sigma)h) \cdot \nabla v \, dx = -\int_\Omega h \nabla u(\sigma) \cdot \nabla v \, dx\,, \qquad \forall\, v \in H_0^1(\Omega)\,. \tag{6}$$

(Hint: Under some smoothness assumption on the domain Ω and the boundary data g (give an example), the solution u is in $H^2(\Omega)$ (regularity)). Generalize (5) to multiple measurement case.

(c) Given the model (1), reconstruct the conductivity σ from the true power density E obtained in step (a) linearizing the problem around σ_0

$$F'(\sigma_0)\sigma = E - F(\sigma_0) + F'(\sigma_0)\sigma_0\,. \tag{7}$$

For ease of derivation and implementation use the so-called "discretize-then-optimize" approach, i.e., work in a finite dimensional setting, discretize the problem then solve it. Note that regularization might be necessary for solving the problem. For implementation, derivatives (5) and (6) can be found in the provided Python code.

(d) Given the model (1), reconstruct the conductivity σ from the true power density E obtained in step (a) using Landweber iteration

$$\sigma_{k+1}^\delta = \sigma_k^\delta + \omega_k^\delta(\sigma_k^\delta) F'(\sigma_k^\delta)^*(E^\delta - F(\sigma_k^\delta))\,, \tag{8}$$

where for the stepsize ω_k^δ one can use the steepest descent stepsize

$$\omega_k^\delta(\sigma) := \frac{\left\|F'(\sigma)^*(E^\delta - F(\sigma))\right\|^2}{\left\|F'(\sigma)F'(\sigma)^*(E^\delta - F(\sigma))\right\|^2}\,. \tag{9}$$

For implementation, derivatives (5) and (6) can be found in the provided Python code. Derive the adjoint of the Fréchet derivative of F. (Hint: first derive it in the L^2 scalar product, i.e., find an operator G such that $\langle F'(\sigma)h, w \rangle_{L^2(\Omega)} = \langle h, G(\sigma)w \rangle_{L^2(\Omega)}$. Then, if time allows, try H^2). Implement Landweber iteration, first, for one measurement of the power density, then generalize it for multiple measurements.

(e) Reconstruct the unknown conductivity σ from the noisy measurements $E^\delta \in L^2(\Omega)$ provided by organizers of the winter school. The data are done using FEniCS and saved in ".mat" files as arrays. Each element of the array corresponds to a coordinate in provided ".xml" mesh file.

References

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