Hybrid Data Tomography: Winter School and Workshop

On Thursday and Friday, January 25–26, 2018, participants of the winter school are offered to work in small groups of 2-3 people on a project from suggested or their own topics. Proposed topics include a short description and a list of ideas to explore. However, participants are not required to answer all suggested questions. Furthermore, the projects are open-ended and participants are encouraged to explore any aspects which appeal to them, with the project descriptions providing ideas. Theoretical or computational studies on the chosen topics should be documented in a short report (5-10 pages) and handed in at the latest on February 2, 2018. For implementing computational tasks, sample codes written in Python using the library FEniCS will be provided at the winter school. To get started with FEniCS, follow instructions at `https://fenicsproject.org/download/`. However, participants are welcome to use any programming language they wish.

Keywords: Inverse Problems, Parameter Identification, Lamé Parameters, Landweber Iteration

Project 3: Quantitative Static Elastography

Let Ω denote a non-empty bounded, open and connected set in $\mathbb{R}^N$, $N = 2$, with a Lipschitz continuous boundary $\partial\Omega$, which has two subsets Γ_D and Γ_T, satisfying $\partial\Omega = \overline{\Gamma_D \cup \Gamma_T}$, $\Gamma_D \cap \Gamma_T = \emptyset$. Given body forces f, displacement g_D, surface traction g_T and Lamé parameters λ and μ, the forward problem of *linearized elasticity* with displacement-traction boundary conditions consists in finding the displacement u satisfying

$$\begin{aligned} -\operatorname{div}(\sigma(u)) &= f \,, \quad \text{in } \Omega \,, \\ u\,|_{\Gamma_D} &= g_D \,, \\ \sigma(u)\vec{n}\,|_{\Gamma_T} &= g_T \,, \end{aligned} \tag{1}$$

where $\vec{n}$ is an outward unit normal vector of $\partial\Omega$ and the *stress tensor* σ defining the stress-strain relation in Ω is defined by

$$\sigma(u) := \lambda \, \operatorname{div}(u)\, I + 2\mu\, \mathcal{E}(u) \,, \qquad \mathcal{E}(u) := \frac{1}{2}\left(\nabla u + \nabla u^T\right) \,, \tag{2}$$

where I is the identity matrix and $\mathcal{E}$ is called the *strain tensor*. Consider the inverse problem of quantitative static elastography, which aims at identifying the unknown Lamé parameters λ, μ from full static displacement field measurements u [2,3].

(a) Download a Python code with the forward solver using the library FEniCS from organizers of the winter school. Simulate the interior data of the displacement field u by computing it as a solution of (1) from $f = 0$, $g_T = 0$, predefined displacement g_D and Lamé parameters λ, μ of your choice.

(b) The inverse problem of quantitative static elastography can be written in the operator form

$$F(\lambda, \mu) = u\,, \tag{3}$$

where the nonlinear operator is introduced by

$$F : D(F) := \left\{(\lambda, \mu) \in H^2(\Omega)^2 |\, \lambda \geq 0, \mu \geq \underline{\mu} > 0\right\} \to L^2(\Omega)^N\,. \tag{4}$$

Assume that operator F is Fréchet differentiable and its derivative $F'(\lambda, \mu)(h_\lambda, h_\mu)$ is given as the solution w of the variational problem

$$\begin{aligned}
&\int_\Omega \left(\lambda \operatorname{div}(w) \operatorname{div}(v) + 2\mu \mathcal{E}(w) : \mathcal{E}(v)\right) dx = \\
&- \int_\Omega \left(h_\lambda \operatorname{div}(u) \operatorname{div}(v) + 2h_\mu \mathcal{E}(u) : \mathcal{E}(v)\right) dx \\
&- \int_\Omega \left(h_\lambda \operatorname{div}(\Phi) \operatorname{div}(v) + 2h_\mu \mathcal{E}(\Phi) : \mathcal{E}(v)\right) dx\,, \qquad \forall\, v \in H^1_{0,\Gamma_D}(\Omega)\,.
\end{aligned} \tag{5}$$

For details, (5) can be found in [3] and provided in the Python code for implementation.

Given the model of linearized elasticity (1), find the Lamé parameters λ, μ from full internal static displacement field u obtained in step (a) linearizing the problem around (λ_0, μ_0)

$$F'(\lambda_0, \mu_0)(\lambda, \mu) = u - F(\lambda_0, \mu_0) + F'(\lambda_0, \mu_0)(\lambda_0, \mu_0)\,. \tag{6}$$

For ease of derivation and implementation use the so-called "discretize-then-optimize" approach, i.e., work in a finite dimensional setting, discretize the problem then solve it. Note that regularization might be necessary for solving the problem.

(c) Recontruct the unknown Lamé parameters λ, μ from the noisy measurement of the displacement field $u^\delta \in L^2(\Omega)^N$ provided by organizers of the winter school. The data are done using FEniCS and saved in ".mat" files as arrays. Each element of the array corresponds to a coordinate in provided ".xml" mesh file.

References

[1] G. S. Alberti and Y. Capdeboscq. Lectures on elliptic methods for hybrid inverse problems. *SAM Report*, 2016.

[2] M. M. Doyley. Model-based elastography: a survey of approaches to the inverse elasticity problem. *Physics in Medicine and Biology*, 57(3):R35–R73, 2012.

[3] S. Hubmer, E. Sherina, A. Neubauer, and O. Scherzer. Lamé Parameter Estimation from Static Displacement Field Measurements in the Framework of Nonlinear Inverse Problems. Submitted. `https://arxiv.org/abs/1710.10446`.